Pick up patty paper and a pencil as you come in,

HOW RIGID MOTIONS CAN CHANGE A GEOMETRY COURSE

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WHY RIGID MOTIONS NOW?

- Rigid Motions are at the foundation of the Common Core approach to geometry at the secondary level.
- An important feature to emphasize is that in the CC model Rigid Motions appear at very early in the geometry course, not as an advanced topic at the end.

WHAT ARE RIGID MOTIONS? – PART 1

- (1) **Rigid Motions of the plane are transformations of the plane.**
- This implies that a rigid motion T is has an inverse T^{-1}, with $T\,T^{-1} = T^{-1}T = I$, the identity map.
- Note: Transformations have been key parts of geometry since the 19th century. In 1923 an important committee of US mathematicians and teachers tried hard to get more transformations into classrooms. So, this is an old story.

WHAT ARE RIGID MOTIONS? – PART 2

- We assume that the plane is provided with a measure $0 \leq |AB|$ of distance between any points A and B so that with this measure lines look like the real number line.
- **(2) Rigid Motions preserve distance.**
- If T is a rigid motion, then always $|T(A)T(B)| = |AB|$.
- Note: A distance-preserving transformation is called an **isometry**.

WHAT ARE RIGID MOTIONS? – PART 3

- We assume the plane is provided with an angle measure $0 \leq \measuredangle ABC \leq 180$ for points B and C distinct from A -- with the usual properties.
- **Rigid Motions preserve angle measure.**
- If T is a rigid motion, then always $\measuredangle T(A)T(B)T(C) = \measuredangle ABC$.
- **IMPORTANT**: This means that rigid motions map lines to lines, since three distinct points A, B, C are collinear if and only if $\measuredangle ABC = 0$ or 180.

LAB: INFORMALLY MODEL RIGID MOTIONS

Take a piece of paper (really or mentally) and think about movements such as rotating the paper or sliding it without turning

- DOING NOTHING: The Identity mapping is a rigid motion.
- DOING AND UNDOING: The inverse of a rigid motion is a rigid motion.
- DOING ONE ACTION FOLLOWED BY A SECOND: The composition of two rigid motions is a rigid motion.

The formal statements can all be proved simply using functional notation.

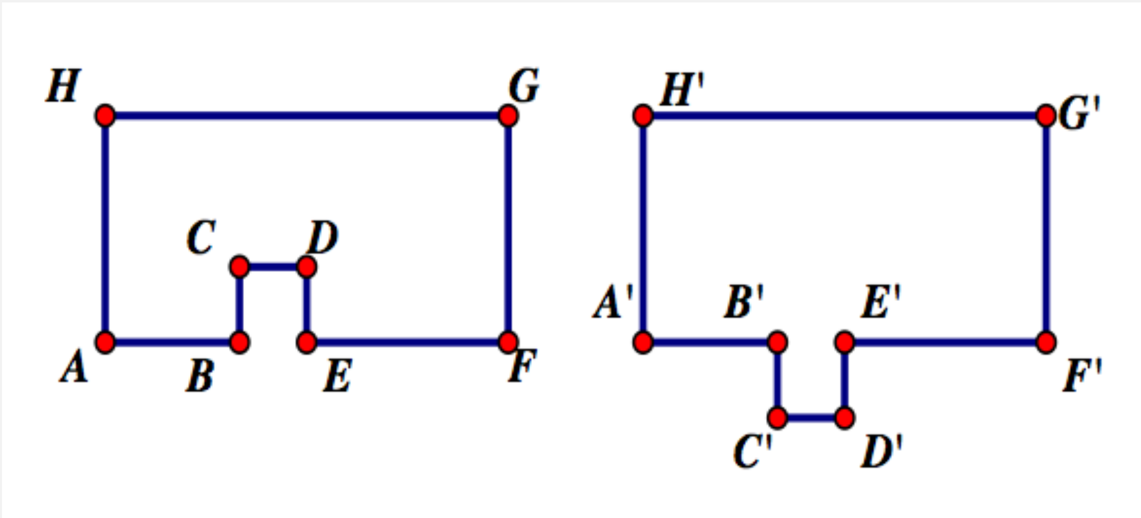
WHAT CAN RIGID MOTIONS DO?

1. Provide a correct and general definition of congruence.
2. Provide a tool to make some proofs simpler, more visual, and more comprehensible.
3. Prove triangle congruence criteria such as SAS instead of assuming SAS as an axiom.
4. Provide tools to prove theorems difficult to prove by other means.

CONGRUENCE!

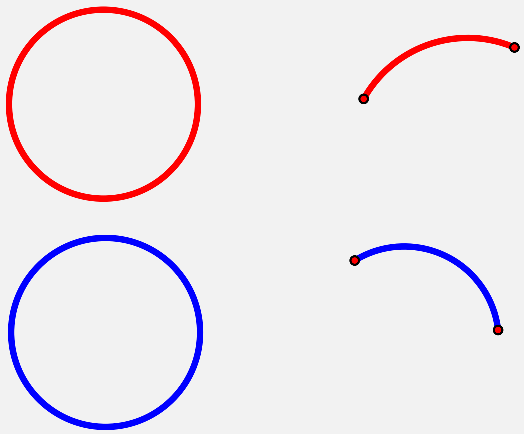
- Commonly textbooks define **congruence** of two figures by saying that corresponding distances and angle measures are the same.
- This works well for triangles and other simple figures, but it has problems in general.
- For example, how does this definition test congruence of two circles? Or lines? Or two ellipses? What distances? What angles? What correspondence? Some examples next:

CORRESPONDING SIDES EQUAL LENGTH; ALL VERTEX ANGLES 90 DEGREES



CONGRUENCE OF FIGURES THAT ARE NOT POLYGONS

- Is the red circle congruent to the blue one?
- Is the red arc congruent to the blue one?
- Are any two lines congruent?



DEFINING CONGRUENCE BY SUPERIMPOSITION RIGOROUSLY

- **Definition: Two figures are congruent if there is a rigid motion that maps one figure onto the other one.**
- Informally, two figures are congruent if you can lay one precisely onto the other, in other words, **to superimpose** one on the other.
- Euclid and the other Greek geometers of antiquity had this intuitive idea too but had no math language for expressing it.
- We do have a language in modern math; we have the language of **functions**.

ADVANTAGES OF THIS DEFINITION

- Can consider congruence of any pair of figures in the plane.
- It is a very general concept. The definition extends to other geometries, such 3-space or space-time) or spherical geometry.
- It makes sense in engineering and science. For instance, one can ask whether two virus particles are congruent.

SOME PATTY PAPER EXPERIMENTS

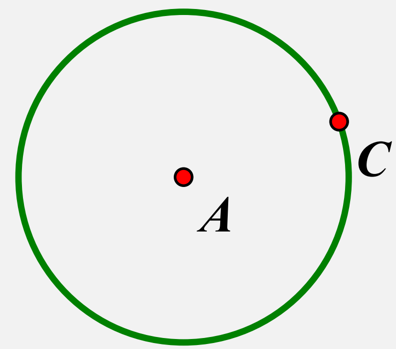
- I suggest that you try to follow the next few slides with patty paper and pencil.

RIGID MOTION WITH ONE FIXED POINT

- To investigate behavior of rigid motions, start with a rigid motion T that fixes a point A, i.e., T(A) = A.
- For any other point C what are the possible images of C?
- Let's experiment with our sheet of patty paper. Mark a point A and a point C. Hold A fixed by pinning it with your pencil point. Move the paper as much as you can. Where can C go?

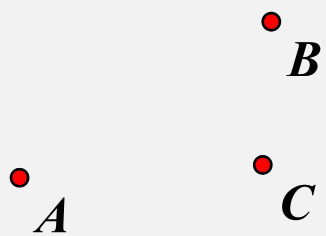
RIGID MOTION WITH ONE FIXED POINT

- The image T(C) must be on the circle with center A that passes through C.
- This is true because T is an isometry. We do not need angle measure for this,



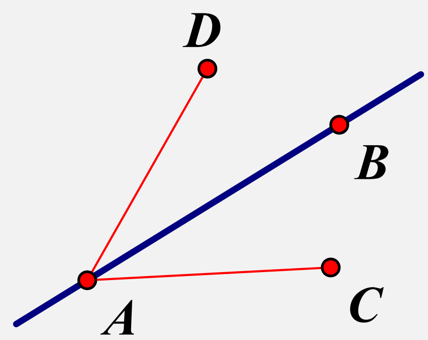
RIGID MOTION WITH TWO FIXED POINTS

- Now suppose a rigid motion T fixes two points A and B, i.e., T(A) = A and T(B) = B.
- For any point C what are the possible images of C?
- Again, experiment with your sheet of patty paper. If you pin both A and B, you will have trouble moving your paper. So, try flipping or FOLDING the paper with the fold through A and B. Mark the point that matches up with C.



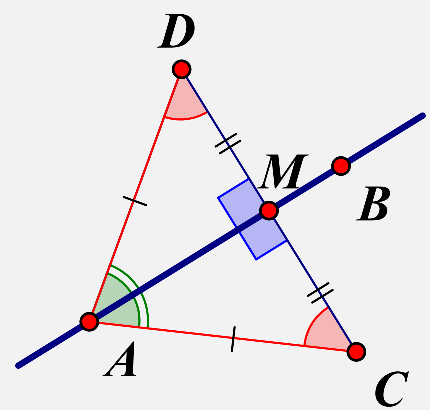
RIGID MOTION WITH TWO FIXED POINTS

- Things to notice when we fold.
- (1) The fold is straight, it's along a line. What is the image of C when C is on the line AB?
- (2) If C is not on this line, there is only one place that D that can be T(C) (besides C itself). To see why, draw equal segments AC and AD with ∡CAB = DAB.



RIGID MOTION WITH TWO FIXED POINTS

- Assume T(C) ≠ C. If we connect points with line segments, we see relationships because T is a rigid motion.
- (a) D is the only possible value for T(C) because ∡CAB = ∡DAB and |DA| = |DC|.
- (b) T(M) = M since M is on the line AB.
- (c) All the pairs of marked angles and segments are congruent!
- (d) line AB is the perpendicular bisector of CD.



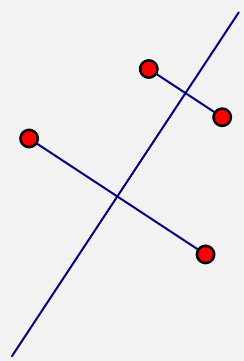
LINE REFLECTION

Let *m* be a line AB. Suppose a transformation T fixes the points of m and for any other point C, the line m is the perpendicular bisector of CT(C).

Then T is called **line reflection in m**, denoted R_m.

We have seen that a rigid motion that fixes the point of m is in fact line reflection in m.

Conversely, we do NOT know that for every m, there is line reflection that is a rigid motion unless there is some axiom that tells us so.

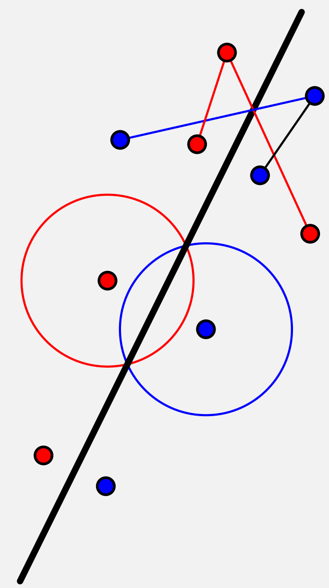


LINE REFLECTION EXISTENCE

Our goal is that for every line m there is a rigid motion that is **reflection in *m*.**

Either

- Assume as an axiom that for every line, there is a rigid motion distinct from the identity that fixes the points of m. (One can prove that this is line reflection as we have done or also add that to the axiom.)
- Assume a different axiom such as Side-Angle-Side for triangles and construct line reflection. The either assume that line reflection is a rigid motion or prove it (a complicated proof for beginners).

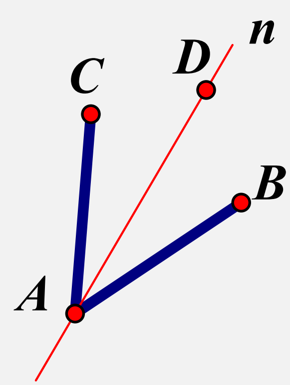


ISOSCELES TRIANGLES

- Demonstrating the basic properties on an isosceles triangle is not very difficult by any approach, but what we have already done offers a quick and direct way to show all the properties at once and in a visual way.

LINE REFLECTION AND ISOSCELES TRIANGLES

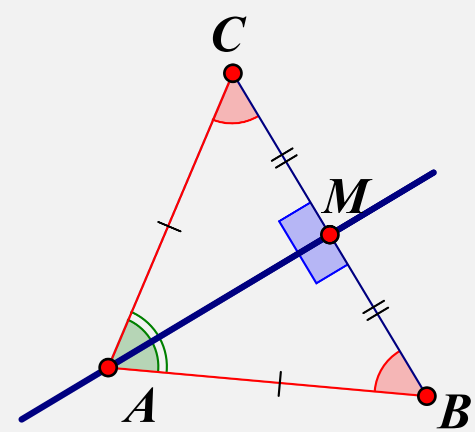
- We start with the sides: |AB| = |AC|. Using angle measure, there is a line n that bisects angle BAC. Let R be reflection in this line.
- Since R is a rigid motion, we can see R(B) = C as before in such a figure.
- Note special case when points are collinear.



REFLECTIONS & ISOSCELES TRIANGLE RELATIONSHIPS

- From this we conclude many things about the isosceles triangle ABC.

1. The bisecting line *n* of angle BAC is the perpendicular bisector of BC.
2. Therefore, for any point P with |PB|=|PC|, then P is on *n*.
3. The base angles ABC and ACB of the triangle are congruent,
4. One can see and feel these relations by folding the figure along *n*.

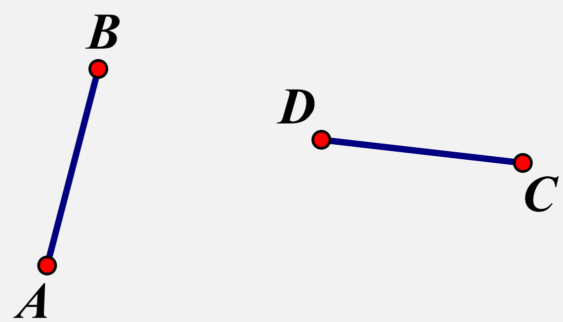


BASIC CONGRUENCE THEOREMS

- We will prove some basic congruence theorems leading up to congruence of triangles. You will follow along by folding your patty paper.
- **Case 0:** Any two points A and B are congruent
- Yes, we need to prove this using the definition! Let m be the perpendicular bisector of segment AB, then reflection in m maps A to B, so the points are congruent.
- Now you can tell your friends that in this session we proved that any two points are congruent!

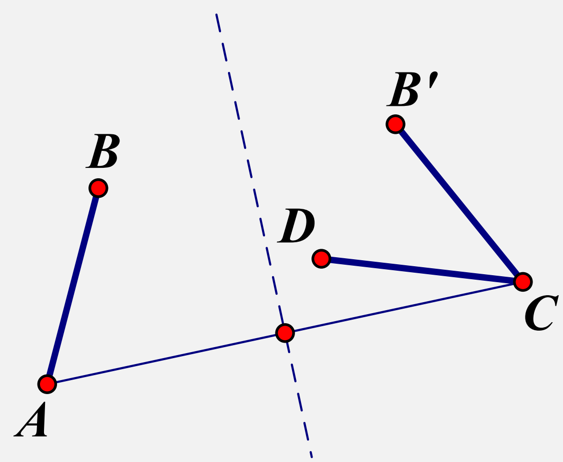
CONGRUENCE OF SEGMENTS

- Suppose AB and CD are segments in the plane of equal length.
- Can we find a rigid motion that takes AB to CD?
- Follow along on a sheet of patty paper if you like.



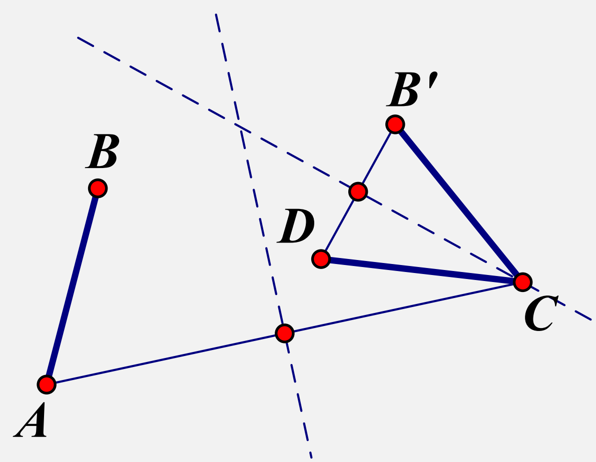
MOVING THINGS: SEGMENTS

- If A = C, triangle ABD is isosceles. Go directly to step 2.
- **Step 1**. Assuming A≠C, reflect AB in the perpendicular bisector of AC. (A is congruent to C!)
- The image of AB is now CB', with |B'C| = |DC|. If B' = D, we are done; stop here.



MOVING THINGS: SEGMENTS

- **Step 2.** In this case, triangle CDB is isosceles. We reflect segment CB' in the perpendicular bisector of B'D. B' reflects to D and C reflects to C.
- Thus, there is a rigid motion that is either one line reflection or the product of two reflections that maps AB to CD.
- Note: We just proved that AB is congruent to CD and that **any two segments of the same length are congruent!**

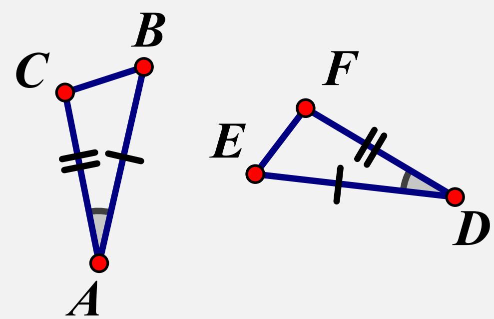


SAS TRIANGLE CONGRUENCE

The Side-Angle-Side (SAS):

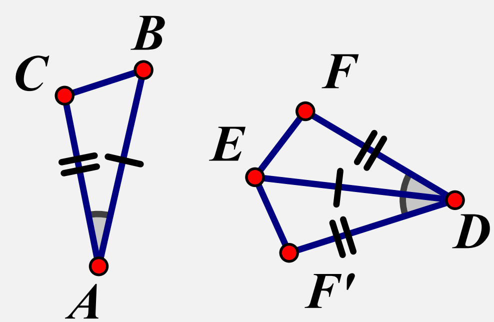
Let ABC and DEF be triangles with |AB|= |DE| and |AC|=|DF| and ∡BAC= ∡EDF. Then the triangles are congruent.

Let's see how to prove this using products of line reflections. Start with a rigid motion T that maps AB to DE.



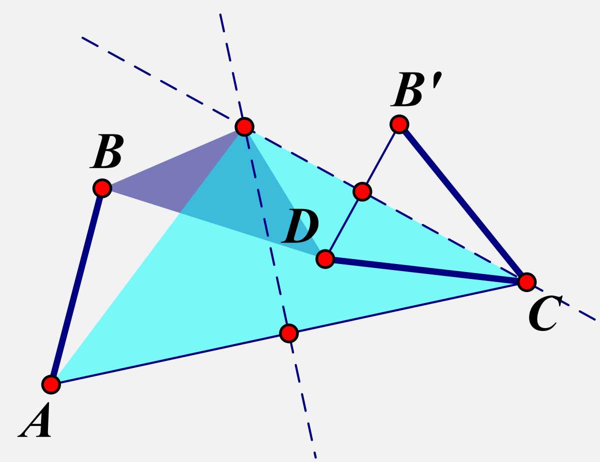
MOVING TRIANGLES USING SAS

- T maps AB to DE, with C' = T(C). Let R be reflection in line DE.
- Since T is rigid, |C'D|= |CA|= |FD| and ∡EDF=∡BAC=∡EDC', so the point C' is either F or F' = R(F).
- If C'=F, T defines a congruence from ABC to DEF; If C'=F', RT does the job.
- So, a product of either 1, 2, or 3 reflections defines a congruence.



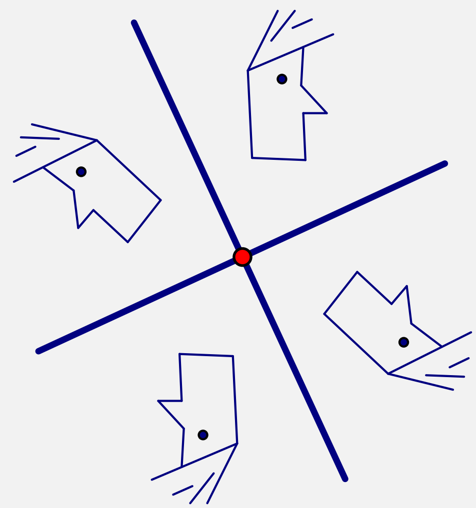
REFLECTIONS ARE OK, BUT WHAT ABOUT THE REST OF THE RIGID MOTIONS?

- The rest of the rigid motions are already here. All of them are products of line reflections.
- ROTATIONS are compositions of reflections in two intersecting lines. The center is the point of intersection. The angle of rotation is double the angle between the lines of reflection. Look at our segment example..
- TRANSLATIONS: The product of two reflections in parallel lines is a translation in the Euclidean plane.
- Rotations are fun to explore with geometry software.



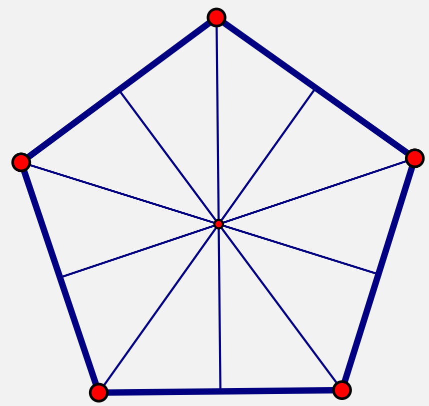
EXAMPLE: HALF-TURN

- The product of reflections in two lines is a rotation with center at the point of intersection and rotation angle twice the angle between the lines.
- For example, if the lines are perpendicular, the rotation angle is 180 degrees and the rotation is called a half-turn or point symmetry.



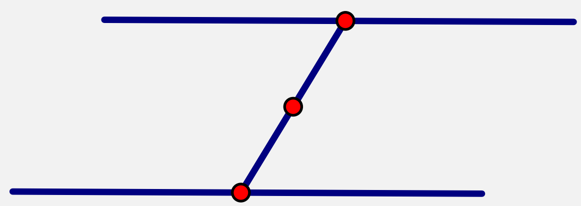
EXAMPLE: REGULAR POLYGONS

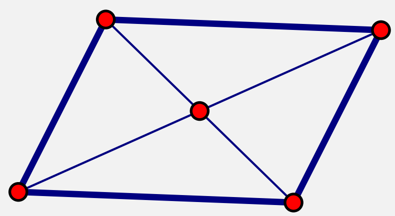
- Regular n-gons have n line reflections that each map the polygons onto itself (i.e., line symmetries).
- The product of any two of these line reflections is a rotation that maps the polygon to itself.



SYMMETRY REASONING: HALF-TURNS

- Two parallel lines and a transversal form a figure with a half-turn symmetry. This rigid motion pairs congruent angles and lengths.
- Also, parallelograms have half-turn symmetry. The center of the rotation s the intersection of the two diagonals.





HOW TO INTRODUCE RIGID MOTIONS INTO A GEOMETRY COURSE

- Here are 3 possible approaches. From the more formal to the more informal.

1. Replace the SAS axiom in your current approach with an axiom that says that for every line, there is a rigid motion distinct from the identity that fixes the points of the line. As soon as possible, prove SAS with this new set of axioms. Once this is proved, you will have everything as usual PLUS rigid motions.
2. Instead of replacing the SAS axiom, just assume the existence of a line reflection for each line as an additional axiom. Now you will have more axioms than you need, but you can reason with them all the same.
3. Although all the other rigid motions are reflection products, just assume they all exist from the start.

A FINE POINT: RIGID MOTIONS VS. ISOMETRIES

- Many books introduce the concept of an isometry. This is a transformation that preserves distance. Nothing is said about angles.
- But – this is tricky – during the development of your geometry course, you can prove that **every isometry is a rigid motion**. The key is **the SSS triangle congruence criterion,** which can be proved by rigid motions. This connects length preservation with angle preservation.
- The tricky part is the order. If you just assume isometry at the outset, you must assume SAS and do a tough proof to show that line reflection is a rigid motion, so this tends to happen later in the course than is optimal.

TRANSFORMATIONS FOR SIMILARITY AND EUCLIDEAN PARALLEL POSTULATE

- The existence of similar figures in the plane at different scale is equivalent to the Euclidean Parallel Postulate.
- One version of EPP is a statement about uniqueness of parallel lines.
- A different option is to introduce this property by a DILATION AXIOM that defines dilation and then says every dilation preserves angles and scales distance.
- From this one can prove the Euclidean parallel property and everything about similarity and similarity transformations.
- There is a talk about this on my website.

RESOURCES FOR LESSONS

- Any treatment of isometries or symmetry in a geometry book, the Math Teacher, or online source.
- H. H. Wu's homepage at UC Berkeley Math Department. Wu was really a motivator for Common Core geometry. This page has links to two of his books that offer examples for teaching with rigid motions.
- Online sources for the Common Core, or new textbooks such as the ones from Illustrative Mathematics.
- The library of geometry explorations for Geogebra.
- Maybe some ideas from my book, *Geometry Transformed*.

ADVICE: RELAX, MAKE SMALL CHANGES

- When talking about Rigid Motions as a kind of revolutionary idea, let's remember that in the end we are talking about the same Euclidean Plane as before, with a few extra tools. The same stuff is true. Once you get through the introduction, all the same proofs will still work.
- There is NO reason to include transformations in every proof, The best style is to use the right tool for each job, You can use several approaches and discuss the differences with your students.
- Don't be overambitious in making changes. It is better to make some modest changes and see how they work out, especially if you are happy with how you already are teaching geometry..

EYES ON THE PRIZE

- The goal is for your students to have a rich understanding of geometrical relationships and how to reason about them.
- There are many paths to get there.

RIGID MOTIONS AND COORDINATES

- If you have coordinates in your plane, some transformations have simple formulas:
- $R_1(x, y) = (x,-y)$ is reflection in the x-axis and $R_2(x,y) = (y,x)$ is reflection in the line $x = y$.
- The product $R_2R_1(x,y) = R_2(x—y) = (-y, x)$ is the formula for rotation by 90 degrees with center (0,0)